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Probability representations of a class of two-way diffusions

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Abstract

There has been little progress in the analysis of two-way diffusion in the last few decades due to the difficulties brought by the interface section similar to a free boundary condition. In this paper, however, the equivalent probability model is considered and the interface section is precisely described by an integral equation. The solution of two-way diffusion is then expressed in an integral form with the integrand being the solution of a classical first passage time model and the solution of a one-dimensional integral equation which is relatively easier to solve. The exact expression of the two-way diffusion enables us to find the explicit solution of the model with infinite horizontal boundaries and without drifting.

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1. Introduction

A counter-current separator is a widely used device to perform chemical separations and is extensively studied in biological and chemical experiments (see, for example, [1, 10]). One of the main purposes of a counter-current separator is to purify a contaminated fluid (gas or liquid). Two different fluids move in opposite directions and the contaminated fluid will gradually become purer. Let ϕ be the dimensionless concentration of the contaminant in the two fluids. Mathematically, the counter-current separator can be described by two-way diffusions, also known as *forward–backward diffusions* or *counter diffusions* [3, 4, 7, 8]. The general model for ϕ is the following partial differential equation (PDE)

$$-h(x)\frac{\partial \phi}{\partial y} = \mu(x)\frac{\partial \phi}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2 \phi}{\partial x^2} \qquad -a \leqslant x \leqslant b \qquad 0 \leqslant y \leqslant 1 \quad (1)$$

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where

$$h(x)\begin{cases}\geqslant 0 & x>0\\ \leqslant 0 & x<0\end{cases}$$

is a smooth function of x for $x \neq 0$, and $h(0) \stackrel{def}{=} 0$, and $\mu, \sigma \in C^1(R)$. The boundary conditions of ϕ are

$$\left.\frac{\partial\phi}{\partial x}\right|_{x=-a+,b-} = 0 \tag{2}$$

and

$$\begin{cases}\phi(x>0, y=1) = 1\\ \phi(x<0, y=0) = 0.\end{cases} \tag{3}$$

Hence at $x = -a$ and $x = b, \phi$ satisfies reflection boundary conditions (equation (2)). Conditions (3) simply imply that one of the phases, say $x < 0$, is pure (see equation (1.1f) in [8]). $\phi(x, y)$ has continuous second-order partial derivatives with respect to x and y except at two points $(x = 0, y = 0)$ and $(x = 0, y = 1)$ where $\phi(x, y)$ is continuous. Hence $h(x)$ describes the opposite speeds in the counter-current device. The application of equation (1) can also be found in other areas such as the distribution of particles impinging upon an elastic point scatterer [7], etc.

Despite the wide applications of model (1), the theoretical developments are far from being satisfactory. Even at a first glance, we could immediately realize the difficulty in solving equation (1) possibly due to the jumping of $h(x)$. Even for the simplest model with the following assumptions:

$$\begin{cases}h(x) = \begin{cases}x^{\alpha-2} & x>0\\ -D|x|^{\alpha-2} & x<0\end{cases} \quad \alpha > 1\\ \mu(x) = 0\\ \sigma^2(x) = 1\end{cases} \tag{4}$$

it is almost impossible to *directly* construct a solution in the framework of well-known techniques for solving a PDE: eigen expansions, due to the lack of orthogonality of the two sets of different eigenfunctions across the two half regions: $\{-a \leqslant x \leqslant 0\}$ and $\{0 \leqslant x \leqslant b\}$, as pointed out in [8]. As a matter of fact, in the situation when $a = b = \infty$, the eigen expansion approach becomes totally invalid. In the circumstances of equation (4), we see that the particle is a purely Brownian particle (without drifting). It moves in opposite directions in the left $x < 0$ and right regions $x > 0$. Furthermore, when $\alpha = 2$, particles in the regions $x < 0$ and $x > 0$ move with constant but opposite directions (if $D > 0$). When $\alpha > 2$, the speed is continuous at $x = 0$. The most interesting case is $1 < \alpha < 2$, the speed is now infinite at $x = 0$, i.e. particles move with opposite and infinite speed at $x = 0+$ and $x = 0-$.

In the current paper, we first develop a general framework, in terms of diffusion processes, to represent the solution of equation (1) with boundary conditions (2) and (3). By virtue of the probability representation, the original problem, to solve a PDE with two opposite speed directions, is replaced by a much simpler problem, to find the solution A of a PDE with one speed and the solution B of an integral equation. The solution of the original equation (1) with boundary conditions (2) and (3) is then obtained via an integration with respect to the two solutions A and B (see exact expression in the next section). To demonstrate the power of the probability representation, we then apply the method to solving the PDE with boundary conditions (4) with $a = b = \infty$. An exact solution is obtained. The exact solution enables us to gain further insights into the problem. For example, the speed $h(x)$ defined by equation (4) is infinite when $\alpha < 2$ and $x \to 0$. We might expect that the solution should exhibit certain singularities, at least at points (0, 0) and (0, 1). However, we find that ϕ is quite well defined (see theorem 3 and figure 2).

2. Probability model

Now let us introduce the equivalent probability model. Consider a particle moving stochastically, the particle's velocity and position at time t are represented by $h(X(t))$ and $Y(t)$, respectively, where $h(x)$ is the function defined above, and $X(t)$ is the diffusion process characterized by

$$\mathrm{d}X(t) = \mu(X(t))\,\mathrm{d}t + \sigma(X(t))\,\mathrm{d}B(t) \tag{5}$$

where μ and σ are functions in $C^1(R^1)$. Therefore

$$Y(t) = \int_0^t h(X(s))\,\mathrm{d}s + Y(0)$$

with a constant $Y(0)$. Let T^0 and T^1 be the first passage times of the particle to the lines $y = 1$ and $y = 0$, namely

$$T^0 = \inf\{t : Y(t) = 0\} \qquad T^1 = \inf\{t : Y(t) = 1\}$$

and

$$\phi(x, y) = \mathrm{Prob}\{T^1 < T^0 | X(0) = x, Y(0) = y\}. \tag{6}$$

Then for the situation with $-a = b = \infty$, we have the following conclusions.

Lemma 1. *The function $\phi(x, y)$ defined by equation (6) satisfies the PDE (1) with boundary conditions (3).*

Proof. With the strong Markovian property and the definition of $\phi(x, y)$, we obtain

$$\begin{aligned}
\phi(x, y) &= \mathrm{Prob}\{T^1 < T^0 | X(0) = x, Y(0) = y\} \\
&= \mathrm{Prob}\{T^1 + \mathrm{d}t < T^0 + \mathrm{d}t | X(0) = x, Y(0) = y\} \\
&= \mathbf{E}\{\mathrm{Prob}\{T^1 + \mathrm{d}t < T^0 + \mathrm{d}t | X(\mathrm{d}t), Y(\mathrm{d}t)\} | X(0) = x, Y(0) = y\} \\
&= \mathbf{E}\{\phi(X(\mathrm{d}t), Y(\mathrm{d}t)) | X(0) = x, Y(0) = y\} \\
&= \phi(x, y) + \frac{\partial \phi}{\partial x}\mathbf{E}(\mathrm{d}X) + \frac{\partial \phi}{\partial y}\mathbf{E}(\mathrm{d}Y) + \frac{1}{2}\frac{\partial^2 \phi}{\partial x^2}\mathbf{E}(\mathrm{d}X)^2 + o(\mathrm{d}t).
\end{aligned}$$

Thus

$$0 = h(x)\frac{\partial \phi}{\partial y} + \mu(x)\frac{\partial \phi}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2 \phi}{\partial x^2}. \tag{7}$$

Also the restrictions

$$\begin{aligned}
&\phi(x \geqslant 0, y = 1) = 1 \\
&\phi(x \leqslant 0, y = 0) = 0
\end{aligned}$$

are obvious from the model's probability meaning. Thus we complete the proof. □

The physical meaning of lemma 1 is quite clear. ϕ is the density of one phase, depending on the travelling speed $h(x)$. When the travelling speed increases, the probability of $T^1 < T^0$ is greater and so the density becomes higher when $y \to 1$. Otherwise, the density becomes lower (see figure 1).

Without loss of generality we assume that $b > a$. For the convenience of later reasoning, we introduce more notation:

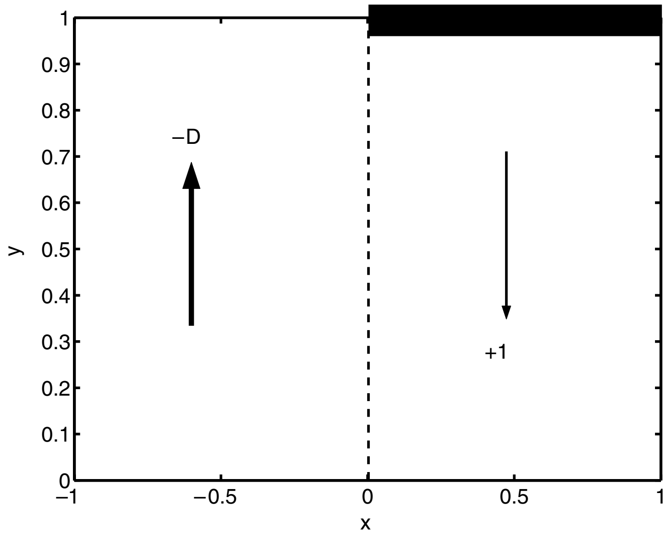

Figure 1. A schematic plot of the two-way diffusion. Particles in the regions $x < 0$ move upwards with a speed $-D = h(x) < 0$, but particles in the regions $x > 0$ move downwards with a speed $h(x) = +1$. The thick horizontal line at $y = 1, x > 0$ represents $\phi(x > 0, y = 1) = 1$. The vertical, dashed line represents the interface between the two phases.

$$T_x = \inf\{t > 0 : X(t) = 0 | X(0) = x > 0\} \tag{8}$$

$$Y_x^* = \int_0^{T_x} h(X(t))\,\mathrm{d}t \tag{9}$$

$$F(x, u) = \mathrm{Prob}\{Y_x^* \leqslant u\} \qquad x > 0 \tag{10}$$

$$F^*(-x, u) : \text{the dual of } F(x, u) \text{ when } x < 0, \text{ i.e. } F^*(-x, u) = F(x, u) \tag{11}$$

$$g(u) = \frac{\mathrm{d}\phi(x = 0, y = u)}{\mathrm{d}u} \tag{12}$$

$$\theta(u) = \frac{\partial F(x = 0+, y = u)}{\partial x} \tag{13}$$

$$\theta^*(u) = -\frac{\partial F^*(-x = 0+, y = u)}{\partial x}. \tag{14}$$

Here we can see T_x is the first impact time of $x = 0$ of the particle, given the initial position x, Y_x^* is the distance travelled by the particle before it hits $x = 0$, and $F(x, u)$ is the distribution function of Y_x^*. Furthermore g and θ are the density functions (derivatives) of ϕ and F. Also to make our formulae tidy we frequently use $\beta = 1/\alpha$.

Lemma 2. *The function $F(x, u)$ is the solution of the following PDE*

$$h(x)\frac{\partial F}{\partial u} = \mu(x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2 F}{\partial x^2} \qquad x > 0 \tag{15}$$

with boundary conditions

$$F(x = 0, u) = 1 \qquad \frac{\partial F(x, u = 0)}{\partial x} = -\delta(x)$$

where $\delta(x)$ is the Dirac function.

Proof. Again using the strong Markovian property and the definition of F, we have

$$F(x,u) = \mathbf{E}(F(x+\mathrm{d}X, u-h(x)\,\mathrm{d}t)|X(0)=x, Y(0)=u)$$
$$= F(x,u) + \frac{\partial F}{\partial x}\mu(x)\,\mathrm{d}t - \frac{\partial F}{\partial u}h(x)\,\mathrm{d}t + \frac{1}{2}\sigma^2(x)\frac{\partial^2 F}{\partial x^2} + o(\mathrm{d}t).$$

Taking the limit for $\mathrm{d}t \to 0$, we have the PDE in the lemma. The boundary conditions are obvious from the definitions. This completes the proof. □

Theorem 1. *For the PDE system (1) with infinite ranges ($a = b = \infty$) and boundary conditions (3), the solution $\phi(x, y)$ could be represented in the following form,*

$$\int_0^y \theta^*(u)g(y-u)\,\mathrm{d}u = \int_0^{(1-y)} \theta(u)g(y+u)\,\mathrm{d}u \tag{16}$$

$$\phi(x,y) = \begin{cases} 1 - \int_0^{(1-y)} F(x,u)g(y+u)\,\mathrm{d}u & x \geqslant 0 \\ \int_0^y F^*(-x,u)g(y-u)\,\mathrm{d}u & x \leqslant 0. \end{cases} \tag{17}$$

Proof. Since the event $\{T^1 < T^0\}$ could be partitioned as the mutually exclusive union of $\{T^1 < T^0, Y_x^* > 1-y\}$ and $\{T^1 < T^0, Y_x^* \leqslant 1-y\}$, with the strong Markovian property of $X(t)$ we have

$$\phi(x,y) = \mathrm{Prob}\{T^1 < T^0|X(0)=x, Y(0)=y\} \tag{18}$$

$$= \begin{cases} \mathrm{Prob}\{Y_x^* > 1-y\} + \int_0^{(1-y)} f_{Y_x^*}(u)\phi(0,y+u)\,\mathrm{d}u & x \geqslant 0 \\ \int_0^y f^*_{Y^*_{-x}}(u)\phi(0,y-u)\,\mathrm{d}u & x \leqslant 0. \end{cases} \tag{19}$$

Performing the integration by parts the second equation of the theorem follows. Then differentiate with respect to x and let x approach $0+$ and $0-$ in two integral forms, respectively; the first equation of the theorem follows immediately. This completes the proof. □

Note here the probability meaning of the interface section leads obviously to the fact that $g(x)$ is a probability density function. We will follow this procedure to analyse the model defined by equations (1) and (4) in the next section.

Theorem 1 is obtained in terms of the diffusion process $X(t)$. However the expressions in theorem 1 are independent of the process $X(t)$. Furthermore we note that ϕ is a linear function of F and F^*. We then conclude that

Corollary 1. *For the PDE system (1) with reflection boundary conditions (2) and (3), the solution $\phi(x, y)$ could be represented as follows,*

$$\int_0^y \theta^*(u)g(y-u)\,\mathrm{d}u = \int_0^{(1-y)} \theta(u)g(y+u)\,\mathrm{d}u \tag{20}$$

$$\phi(x,y) = \begin{cases} 1 - \int_0^{(1-y)} F(x,u)g(y+u)\,\mathrm{d}u & x \geqslant 0 \\ \int_0^y F^*(-x,u)g(y-u)\,\mathrm{d}u & x \leqslant 0 \end{cases} \tag{21}$$

where F is defined in lemma 2 with $\frac{\partial F(x,u=b-)}{\partial x} = 0$ and F^* with $\frac{\partial F^*(x,u=-a+)}{\partial x} = 0$.

Clearly theorem 1 and corollary 1 provide us with a general framework to solve the two-way diffusions:

- *Step 1*. Find the function $F(x, u)$ by solving the PDE (15) in lemma 2. Note that $F(x, u)$ is restricted to the regions with $x > 0$ and is much easier to solve analytically than equation (1).
- *Step 2*. Find function $g(u)$ by solving the integral equation in theorem 1.
- *Step 3*. The solution $\phi(x, y)$ can then be obtained in the integral form as in theorem 1 or corollary 1.

3. Explicit solution

As a demonstration of the applications of our theory developed before, in this section we perform a fairly exclusive analysis of the two-way diffusion model with conditions as in equation (4) defined in section 1. The range of x is the whole real axis, i.e. the boundaries are at infinity.

With $\beta = 1/\alpha$, we have the following theorem.

Theorem 2. *Suppose* $Z \sim \Gamma(\beta)$, $x > 0$, $u > 0$, *and*

$$G(x, u) = Prob\left\{Z > \frac{2\beta^2 x^\alpha}{u}\right\} = \int_{\frac{2\beta^2 x^\alpha}{u}}^{\infty} \frac{z^{\beta-1}\,\mathrm{e}^{-z}}{\Gamma(\beta)}\,\mathrm{d}z. \tag{22}$$

Then

$$\frac{1}{2}\frac{\partial^2 G}{\partial x^2} = x^{\alpha-2}\frac{\partial G}{\partial u} \tag{23}$$

$$G(x = 0, u) = 1 \qquad \left.\frac{\partial G}{\partial x}\right|_{u=0} = -\delta(x) \tag{24}$$

where $\delta(x, 0)$ *is the Dirac delta function at 0.*

So $G(x, u) = F(x, u)|_{x>0}$ *(defined in lemma 2).*

Proof. It is readily seen that

$$\frac{\partial G}{\partial u} = \frac{(2\beta^2)^\beta x}{u^{\beta+1}}\,\mathrm{e}^{-\frac{(2\beta^2)}{u}} \Big/ \Gamma(\beta) \tag{25}$$

$$\frac{\partial G}{\partial x} = -\frac{\alpha(2\beta^2)^\beta}{u^\beta}\,\mathrm{e}^{-\frac{(2\beta^2)x}{u}} \Big/ \Gamma(\beta) \tag{26}$$

$$\frac{\partial^2 G}{\partial x^2} = \frac{\alpha^2(2\beta^2)^{\beta+1}x^{\alpha-1}}{u^{\beta+1}}\,\mathrm{e}^{-\frac{(2\beta^2)x}{u}} \Big/ \Gamma(\beta) \tag{27}$$

$$= \frac{2(2\beta^2)^\beta x^{\alpha-1}}{u^{\beta+1}}\,\mathrm{e}^{-\frac{(2\beta^2)x}{u}} \Big/ \Gamma(\beta). \tag{28}$$

Therefore,

$$\frac{1}{2}\frac{\partial^2 G}{\partial x^2} = x^{\alpha-2}\frac{\partial G}{\partial u}. \tag{29}$$

The boundary conditions are obvious. The proof is completed. □

Lemma 3. *The interface section curve* $g(y) = \partial\phi(x = 0, y)/\partial y$, *i.e. the solution of the integral equation (16), is the* $\mathrm{B}(\gamma, \delta)$ *density function,*

$$g(y) = \frac{1}{\mathrm{B}(\gamma, \delta)} y^{\gamma-1}(1 - y)^{\delta-1} \tag{30}$$

where

$$\delta = \frac{1}{\pi}\arctan\left(\frac{\sin(\pi\beta)}{D^{\beta} + \cos(\pi\beta)}\right) \tag{31}$$

$$\gamma = \beta - \delta \tag{32}$$

$$= \frac{1}{\pi}\arctan\left(\frac{\sin(\pi\beta)}{D^{-\beta} + \cos(\pi\beta)}\right) \tag{33}$$

$$\beta = 1/\alpha. \tag{34}$$

Proof. From theorem 1,

$$\phi(x, y) = \begin{cases} 1 - \int_0^{(1-y)} F(x, u) g(y+u)\,\mathrm{d}u & x \geqslant 0 \\ \int_0^{y} F^*(-x, u) g(y-u)\,\mathrm{d}u & x \leqslant 0. \end{cases}$$

Note that the particular form of function $h(x)$ leads to

$$F^*(-x, u) = F\left(-x, \frac{u}{D}\right) \qquad x < 0 \tag{35}$$

and

$$\theta^*(u) = \theta\left(\frac{u}{D}\right). \tag{36}$$

Also theorem 3 leads to $\theta(u) \propto u^{-\beta}$. So the integral equation in theorem 1 becomes

$$D^{\beta}\int_0^{y} u^{-\beta} g(y-u)\,\mathrm{d}u = \int_0^{1-y} u^{-\beta} g(y+u)\,\mathrm{d}u. \tag{37}$$

Let

$$g(u) = \frac{1}{\mathrm{B}(\gamma, \delta)} u^{\gamma-1}(1-u)^{\delta-1} \qquad (\gamma + \delta = \beta). \tag{38}$$

We intend to determine the parameters γ and δ to satisfy equation (37). With the identity

$$\int_0^1 x^{\delta-1}(1-x)^{\gamma-1}(1+ax)^{-(\delta+\gamma)}\,\mathrm{d}x = (1+a)^{-\delta} B(\gamma, \delta) \tag{39}$$

the left- and right-hand sides of (37) become

$$\text{LHS} = D^{\beta}\frac{B(1-\beta, \gamma)}{B(\gamma, \delta)} y^{-\delta}(1-y)^{-\gamma} \tag{40}$$

$$\text{RHS} = \frac{B(1-\beta, \delta)}{B(\gamma, \delta)} y^{-\delta}(1-y)^{-\gamma}. \tag{41}$$

Letting LHS = RHS, we have

$$\begin{aligned} D^{\beta} &= \frac{B(1-\beta, \delta)}{B(1-\beta, \gamma)} \\ &= \frac{\Gamma(1-\beta)\Gamma(\delta)}{\Gamma(1-\beta+\delta)} \frac{\Gamma(1-\beta+\gamma)}{\Gamma(1-\beta)\Gamma(\gamma)} \\ &= \frac{\Gamma(\delta)\Gamma(1-\delta)}{\Gamma(\gamma)\Gamma(1-\gamma)} = \frac{\pi}{\sin(\pi\delta)} \frac{\sin(\pi\gamma)}{\pi} \\ &= \frac{\sin(\pi(\beta-\delta))}{\sin(\pi\delta)} = \frac{\sin(\pi\beta)\cos(\pi\delta) - \sin(\pi\delta)\cos(\pi\beta)}{\sin(\pi\delta)} \\ &= \sin(\pi\beta)\cot(\pi\delta) - \cos(\pi\beta) \end{aligned}$$

$$\cot(\pi\delta) = \frac{D^{\beta} + \cos(\pi\beta)}{\sin(\pi\beta)}.$$

Thus

$$\delta = \frac{1}{\pi} \arctan\left(\frac{\sin(\pi\beta)}{D^{\beta} + \cos(\pi\beta)}\right) \tag{42}$$

$$\gamma = \beta - \delta. \tag{43}$$

This completes our proof. □

Summarizing all the above results concerning the model, we have

Theorem 3. *The model has the explicit solution in the following integral form,*

$$F(x, u) = \int_{\frac{2\beta^2 x^{\alpha}}{u}}^{\infty} \frac{z^{\beta-1}\,\mathrm{e}^{-z}}{\Gamma(\beta)}\,\mathrm{d}z \qquad g(y) = \frac{1}{B(\gamma, \delta)} y^{\gamma-1}(1-y)^{\delta-1}$$

where

$$\delta = \frac{1}{\pi} \arctan\left(\frac{\sin(\pi\beta)}{D^{\beta} + \cos(\pi\beta)}\right)$$

$$\gamma = \beta - \delta$$

$$\phi(x, y) = \begin{cases} 1 - \int_0^{(1-y)} F(x, u) g(y+u)\,\mathrm{d}u & x \geqslant 0 \\ \int_0^{y} F\left(-x, \frac{u}{D}\right) g(y-u)\,\mathrm{d}u & x \leqslant 0. \end{cases}$$

Byproduct. From the symmetry of γ and δ in the solution of the model, we immediately have the following triangular identity which is easy to verify but not obvious by sight.

Corollary 2. *For any* $\beta > 0$ *and* $C > 0$,

$$1 = \frac{1}{\pi\beta} \arctan\left(\frac{\sin(\pi\beta)}{C + \cos(\pi\beta)}\right) + \frac{1}{\pi\beta} \arctan\left(\frac{\sin(\pi\beta)}{C^{-1} + \cos(\pi\beta)}\right). \tag{44}$$

When $C = 1$ this is exactly the half-angle formula.

There are two 'bending points' on the solution surface $(x, y, \phi(x, y))$ where the function $\frac{\partial\phi}{\partial x}$ is not continuous. They are $(0, 0, \phi(0, 0))$ and $(0, 1, \phi(0, 1))$. We consider the behaviour of $\phi(x, y)$ near $(x = 0+, y = 0)$ only. The other situation could be considered similarly. For the model we have the following result.

Corollary 3. *For* $x > 0$ *and* $y = 0$,

$$\phi(x, y = 0) = \int_0^{2\beta^2 x^{\alpha}} \frac{t^{\gamma-1}\,\mathrm{e}^{-t}}{\Gamma(\gamma)}\,\mathrm{d}t \tag{45}$$

and

$$\phi(x, y = 0) \approx \frac{(2\beta^2)^{\gamma}}{\Gamma(\gamma+1)} x^{\eta} \qquad \textit{when } x \textit{ is close to } 0 \tag{46}$$

where

$$\eta = \frac{1}{\pi\beta} \arctan\left(\frac{\sin(\pi\beta)}{D^{-\beta} + \cos(\pi\beta)}\right). \tag{47}$$

So we have the range of η,

$$\lim_{D\uparrow\infty} \eta = 1 \tag{48}$$

$$\eta = 1/2 \qquad \textit{when} \quad D = 1 \tag{49}$$

$$\lim_{D\downarrow 0} \eta = 0. \tag{50}$$

Proof. From theorem 3 we have, for $x > 0$,

$$\phi(x, y = 0) = 1 - \int_0^1 F(x, u) g(u)\, \mathrm{d}u \tag{51}$$

$$= 1 - \int_0^1 \mathrm{Prob}\left\{ Z \geqslant \frac{2\beta^2 x^\alpha}{u} \right\} g(u)\, \mathrm{d}u \tag{52}$$

$$= \int_0^1 \mathrm{Prob}\left\{ Z \leqslant \frac{2\beta^2 x^\alpha}{u} \right\} g(u)\, \mathrm{d}u \tag{53}$$

$$= \mathrm{Prob}\{ZU \leqslant 2\beta^2 x^\alpha\} \tag{54}$$

where $Z \sim \Gamma(\beta)$ and $U \sim \mathrm{B}(\gamma, \delta)$ are two independent random variables naturally derived from previous lemmas and theorems.

Since $\beta = \gamma + \delta$, we have $ZU \sim \Gamma(\gamma)$. So

$$\phi(x > 0, y = 0) = \int_0^{2\beta^2 x^\alpha} \frac{t^{\gamma-1}\, \mathrm{e}^{-t}}{\Gamma(\gamma)}\, \mathrm{d}t. \tag{55}$$

With $x \sim 0+$,

$$\phi(x > 0, y = 0) \approx \frac{1}{\Gamma(\gamma)} \int_0^{2\beta^2 x^\alpha} t^{\gamma-1}\, \mathrm{d}t \tag{56}$$

$$= \frac{(2\beta^2)^\gamma}{\Gamma(\gamma + 1)} x^{\alpha\gamma} \tag{57}$$

as we expected ($\eta = \alpha\gamma$).

The range of η is obvious. Thus we complete the proof. □

From corollary 3, we see that $\phi(x, 0)$ approaches zero when $x \to 0$ and $D > 0$, i.e. at point $x = 0$, $y = 0$ there are no particles left. Nevertheless, when $D = 0$, there is no flow of count currents, we see that $\phi(x, 0)$ is a constant. More exactly it equals its initial value.

In figure 2 we plot ϕ versus (x, y) with $\alpha = 1.5, 2, 3$. It is easily seen that the smaller α is, the sharper the changes of ϕ (comparing the upper panel (right) with the bottom panel (right)).

4. Discussions

In terms of diffusion processes, we first found an integral expression for the two-way diffusion. The difficulty of solving the two-way diffusion is reduced to the problem of solving the usual diffusion equation and an integral equation. The approach enables us to find the exact solution of the model without drifting and with infinite boundary conditions. From the exact solution, we got a much better understanding of the properties of the solution. To the best of our knowledge, such an exact solution has not been reported in the literature.

When $\alpha \in (0, 1]$, the solution of the first passage model still exists uniquely as equation (22). But no equilibrium could be reached in our probability model. In other words, there is no solution for the integral equation (16), so the corresponding two-way diffusion problem is not well posed. Geometrically, when α approaches 1+, $\frac{\partial \phi}{\partial x}$ approaches infinity at $x = 0$. On the other hand, this implies that the interface section functions of all the situations (with infinite boundaries) are the set of all the Beta distributions with the sum of the parameters being smaller than 1.

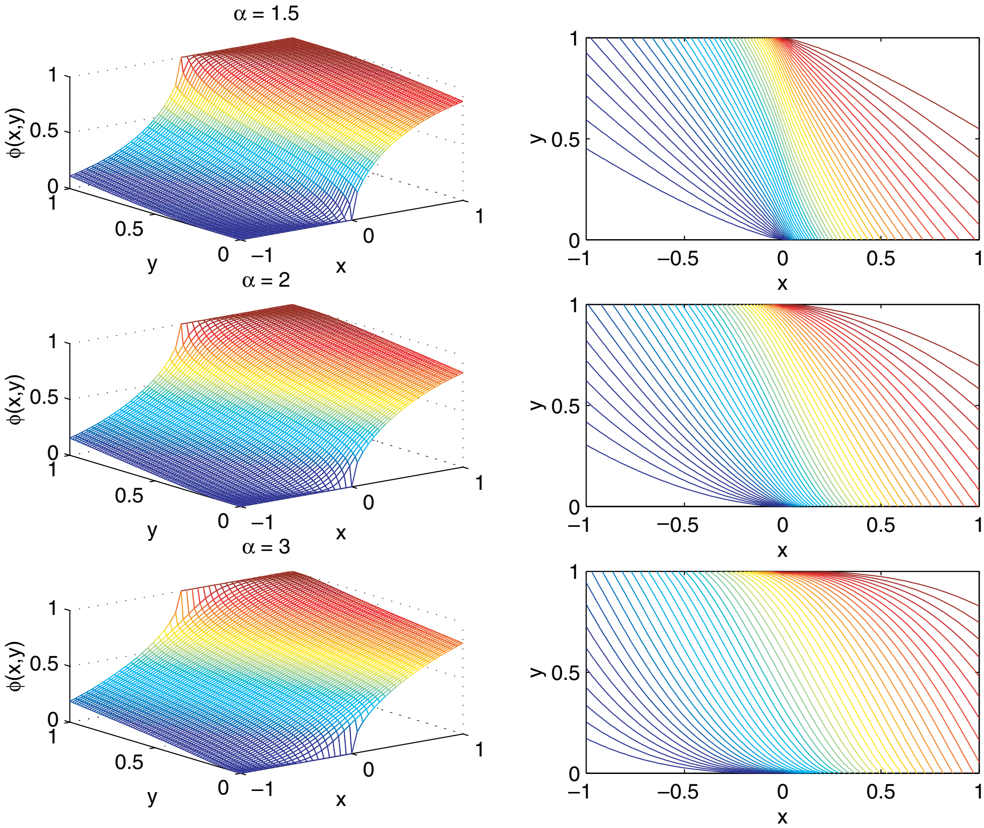

Figure 2. $\phi(x, y)$ versus (x, y) with $\alpha = 1.5, 2, 3$ (left). Contour plots of $\phi(x, y)$ (right). It is easily seen that the interfaces between left regions $x < 0$ and right regions $x > 0$ become sharper when α decreases.

Furthermore, from the exact solution, we can ask why the singularity in the speed $(1 < \alpha < 2)$ does not result in a singularity in the solution. To answer this question we need to investigate the properties of stopping times defined in this paper, which is one of our ongoing research topics.

In summary, in addition to the half-range eigen expansion schemes [8], the integral equation describing the interface section and developed in the current paper offers another powerful and promising tool to explore the model.

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